On the Bound of the Number of the Real Roots of a Random Algebraic Polynomial(Nonlinear Analysis and Convex Analysis)

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確率代数多項式の実数根の個数の限界について

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1 Introduction

A random algebraic polynomial of degree n is of the form

$$F_n(x,\omega)=\sum_{k=0}^{n}a_k(\omega)x^k,$$

where the $a_k(\omega)$ are random variables and x is a complex number. Since Bloch and Polya[1] initiated the estimate of the number of real roots of a random algebraic polynomial, there has been a stream of papers on the various estimates of the zeros of random algebraic polynomials by others, like Littlewood & Offord[3] and Evans[2], although they mainly work with independent and identically distributed coefficients. For dependent coefficients, Sambandham[4] obtained asymptotic formulae for the expectation of the number of real roots of a random algebraic polynomial in the case of random coefficients are normally distributed with mean zero, variance 1 and each correlation $\rho_{ij}=\rho\in(0,1)$ or $\rho^{|i-j|},\rho\in(0,\frac{1}{2})$. Also for the upper bound of the number of real roots of a random algebraic polynomial, Sambandham[5] considered the case of constant correlation $\rho\in(0,1)$.

We have researched the estimate with respect to the upper and lower bounds of the number of real roots of a random algebraic polynomial whose coefficients are dependent normal random variables with varying correlation.

2 Upper Bound of the Number of Real Roots

First we suppose that the coefficients are normally distributed random variables having mean zero, variance 1 and each correlation $\rho_{ij}=\rho_{|i-j|}$, where $\{\rho_k\}$ is a nonnegative decreasing sequence satisfying $\rho_1<\frac{1}{2}$ and $\sum_{k=1}^{\infty}\rho_k<\infty$. That is to say that we consider the random coefficients $a_k(\omega)$ $k=0,1,\cdots,n$ have joint density function

$$|M|^{\frac{1}{2}}(2\pi)^{-\frac{n+1}{2}}\exp\left(-\frac{1}{2}\boldsymbol{a}'M\boldsymbol{a}\right),$$

where M^{-1} is the moment matrix with

$$\rho_{ij} = \begin{cases} 1 & (i = j) \\ \rho_{|i-j|} & (i \neq j) \end{cases}$$

where $\{\rho_j\}$ is a nonegative decreasing sequence satisfying $\rho_1 < \frac{1}{2}$ and $\sum_{j=1}^{\infty} \rho_j < \infty$. $\boldsymbol{a}'$ is the transpose of the column vector $\boldsymbol{a}$.

Theorem 1 ([6]). *There exists an integer n_0 such that for each $n > n_0$,the number of real roots of the equations $F_n(z, \omega) = 0$ is at most*

$$C(\log \log n)^2 \log n$$

except for a set of measure at most

$$\frac{C'}{\log n_0 - \log \log \log n_0},$$

where C and C' are constants.

Proof. We indicate a brief outline of the proofs. We must remark that the transformation $x \to \frac{1}{x}$ makes the equation $F_n(x, \omega) = 0$ transformed to $\sum_{r=0}^{n} a_{n-r}(\omega) x^r = 0$ and $(a_0(\omega), a_1(\omega), \cdots, a_n(\omega))$ and $(a_n(\omega), a_{n-1}(\omega), \cdots, a_0(\omega))$ have the same joint density function. Therefore the number of roots and the measure of the exceptional set in the range $[-\infty, \infty]$ are twice the corresponding estimates for the range $[-1, 1]$. But we consider the range $[-1, 0]$ only. Because it can be shown that the upper bound in $[0, 1]$ is the same as in $[-1, 0]$ by using the same procedure. Thus the number of roots in the range $[-\infty, \infty]$ and the measure of the exceptional set are each four times the corresponding estimates for the range $[-1, 0]$.

The proof consists of defining circles to cover the interval $[0, 1]$ and estimating the number of zeros in each circle by the inequality proved by Jensen's theorem. Let $N(|z - z_0| < r)$ be the number of zeros of a regular function $\phi(z)$ in the circle with center z_0 and of radius r. The following is the inequality essential in order to get the theorem,

$$N(|z - z_0| < r) \leq \frac{\log \left(\frac{\sup_{|z-z_0|<R} |\phi(z)|}{|\phi(z_0)|} \right)}{\log(R/r)}$$

where $R(> r)$.

3 Lower Bound of the Number of Real Roots

Consider

$$f_n(x,\omega) = \sum_{k=0}^{n} a_k(\omega) b_k x^k,$$

where the b_k are positive numbers and the coefficients be m-dependent stationary Gaussian random variables with mean zero and variance 1. In other words, we assume the random coefficients $a_k(\omega)$ $k = 0, 1, \cdots, n$ have joint density function

$$|M|^{\frac{1}{2}} (2\pi)^{-\frac{n+1}{2}} \exp\left(-\frac{1}{2}\boldsymbol{a}'M\boldsymbol{a}\right),$$

where M^{-1} is the moment matrix with

$$\rho_{ij} = \begin{cases} 1 & (i=j) \\ \rho_{|i-j|} \in [0,1) & (1 \le |i-j| \le m) \\ 0 & (|i-j| > m) \qquad i,j = 0,1,\cdots,n \end{cases}$$

Under the above condition we get the following results.

Theorem 2 ([7]). *Let b_k, $k = 0, 1, \cdots, n$ be positive numbers such that*

$$\frac{k_n}{t_n} = o(\log n), \quad \textit{where } k_n = \max_{0\le k\le n} b_k \quad \textit{and} \quad t_n = \min_{0\le k\le n} b_k.$$

Then for $n > n_0$, the number of real roots of the equations $f_n(x,\omega) = 0$ is at least

$$\frac{C\log n}{\log\left(\frac{k_n}{t_n}\log n\right)}$$

except for a set of measure at most

$$\frac{C'\log\left(\frac{k_n}{t_n}\log n\right)}{\log n}$$

where C, C' are positive constants.

Proof. The method of the proof consists mainly of counting the number of crossing in each interval of length δ.

As the improvement of theorem 2, we get the following estimate.

Theorem 3. *Let b_k, $k = 0, 1, \cdots, n$ be positive numbers such that $\lim_{n\to\infty} \frac{k_n}{t_n}$ is finite, where*

$$k_n = \max_{0\le k\le n} b_k \quad \textit{and} \quad t_n = \min_{0\le k\le n} b_k.$$

Then for $n > n_0$, the number of real roots of most of the equations $f_n(x,\omega) = 0$ is at least

$$\epsilon_n \log n$$

except for a set of measure at most

$$\frac{C}{\epsilon_n \log n} + \left(\frac{k_n}{t_n}\right)^{\beta} \exp\left(-\frac{C'\beta}{\epsilon_n}\right), \beta > 0,$$

provided ϵ_n *tends to zero but* $\epsilon_n \log n$ *tends to infinity as* n *tends to infinity, where* C *and* C' *are positive constants.*

Proof. We borrow the method of the proof of theorem 2.

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